# Classes of behavior of small-world networks

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Small-world networks are the focus of recent interest because they appear to circumvent many of the limitations of either random networks or regular lattices as frameworks for the study of interaction networks of complex systems. Here, we report an empirical study of the statistical properties of a variety of diverse real-world networks. We present evidence of the occurrence of three classes of small-world networks: (a) *scale-free* networks, characterized by a vertex connectivity distribution that decays as a power law; (b) *broad-scale* networks, characterized by a connectivity distribution that has a power-law regime followed by a sharp cut-off; (c) *single-scale* networks, characterized by a connectivity distribution with a fast decaying tail. Moreover, we note for the classes of *broad-scale* and *single-scale* networks that there are constraints limiting the addition of new links. Our results suggest that the nature of such constraints may be the controlling factor for the emergence of different classes of networks.

Disordered networks, such as small-world networks are the focus of recent interest because of their potential as models for the interaction networks of complex systems [1–6]. Specifically, neither random networks nor regular lattices appear to be an adequate framework within which to study "real-world" complex systems [7] such as chemical-reaction networks [8], neuronal networks [1], food-webs [9–11], social networks [12–14], scientific-collaboration networks [15], and computer networks [4,16,17].

Small-world networks [1,2] —which emerge as the result of randomly replacing a fraction $p$ of the links of a $d$-dimensional lattice with new random links— interpolate between the two limiting cases of a regular lattice ($p = 0$) and a random graph ($p = 1$). A "small-world" network is characterized by the properties (i) the local neighborhood is preserved —as for regular lattices [1,2]—, and (ii) the diameter of the network, quantified by average shortest distance between two vertices [18], increases logarithmically with the number of vertices $n$ —as for random graphs [19]. The latter property gives the name "small-world" to these networks, as it is possible to connect any two vertices in the network through just a few links while the local connectivity would suggest the network to be of finite dimensionality.

The structure of small-world networks and of real networks has been probed through the calculation of their diameter as a function of network size [1]. In particular, networks such as (a) the electric-power grid for Southern California, (b) the network of movie-actor collaborations, and (c) the neuronal network of the worm *C. Elegans*, appear to be small-world networks [1]. Further, it was proposed [5] that these three networks, the world-wide web [4], and the network of citations of scientific papers [20,21] are scale-free —that is, they have a distribution of connectivities that decays with a power-law tail.

Scale-free networks emerge in the context of a growing network in which new vertices connect preferentially to the more highly connected vertices in the network [5]. Scale free networks are still small-world networks because (i) they have clustering coefficients much larger than random networks [1,2], and (ii) their diameter increases logarithmically with the number of vertices $n$ [5].

Here, we address the question of the conditions under which disordered networks are scale-free through the analysis of several networks in social, economic, technologic, biologic, and physical systems. We identify a number of systems for which there is a single scale for the connectivity of the vertices. For all these networks there are constraints limiting the addition of new links. Our results suggest that such constraints may be the controlling factor for the emergence of scale-free networks.

First, we consider two examples of technologic and economic networks: (i) the electric-power grid of Southern California [1], the vertices being generators, transformers and substations and the links high-voltage transmission lines, and (ii) the network of world airports [22], the vertices being the airports and the links non-stop connections. Figure 1 shows the connectivity distribution for these two examples. It is visually apparent that neither case has a power-law regime, and that both have exponentially decaying tails, implying that there is a single scale for the connectivity $k$.

Second, we consider three examples of "social" networks: (iii) the movie-actors network [1], the links in this network indicating that the two actors were cast at least once in the same movie, (iv) the acquaintance network of Mormons [23], the vertices being 43 Utah Mormons and the number of links the number of other Mormons they know, and (v) the friendship-network of 417 Madison Junior High School students [24]. Figure 2 shows the connectivity distribution for these social networks. The scale-free (power-law) behavior of the actor's network [5] is truncated by an exponential tail. In contrast, the network of acquaintances of the Utah Mormons and the friendship network of the high-school students dis-

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play no power-law regime, but instead we find results consistent with a Gaussian distribution of connectivities, indicating the existence of a single scale for $k$.

Third, we consider two examples of networks from the natural sciences: (vi) the neuronal network of the worm *C. Elegans* [1,25,26], the vertices being the neurons and the links being connections between neurons, and (vii) the conformation space of a lattice polymer chain [27], the vertices being the possible conformations of the polymer chain and the links the possibility of connecting two conformations through local movements of the chain [27]. The conformation space of a protein chain shares many of the properties [27] of the small-world networks of Ref. [1]. Figures 3a,b show for *C. Elegans* the cumulative distribution of $k$ for both incoming and outgoing neuronal links. The tails of both distributions are well approximated by exponential decays, consistent with a single scale for the connectivities. For the network of conformations of a polymer chain the connectivity follows a binomial distribution, which converges to the Gaussian [27], so we also find a single scale for the connectivity of the vertices (Fig. 3c).

Thus, there is empirical evidence for the occurrence of three classes of small-world networks: (a) *scale-free* networks, characterized by a connectivity distribution with a tail that decays as a power law [4,20,21]; (b) *broad-scale* or truncated scale-free networks, characterized by a connectivity distribution that has a power-law regime followed by a sharp cut-off, like an exponential or Gaussian decay of the tail [see example (iii)]; (c) *single-scale* networks, characterized by a connectivity distribution with a fast decaying tail, such as exponential or Gaussian [see examples (i),(ii),(iv-vii)].

A natural question is "What are the reasons for such a rich range of possible structures for small-world networks?" To answer this question let us recall that preferential attachment in growing networks gives rise to a power-law distribution of connectivities [5]. However, preferential attachment can be hindered by two classes of factors: (I) *aging* of the vertices. This effect can be pictured for the network of actors: in time, every actress or actor will stop acting. For the network, this fact implies that even a very highly connected vertex will, eventually, stop receiving new links. The vertex is still part of the network and contributing to network statistics, but it no longer receives links. The aging of the vertices thus limits the preferential attachment preventing a scale-free distribution of connectivities. (II) *cost* of adding links to the vertices or the limited *capacity* of a vertex. This effect is exemplified by the network of world airports: for reasons of efficiency, commercial airlines prefer to have a small number of hubs where all routes would connect. To first approximation, this is indeed what happens for individual airlines, but when we consider all airlines together, it becomes physically impossible for an airport to become a hub to all airlines. Due to space and time constraints, each airport will limit the number of landings/departures per hour, and the number of passengers in transit. Hence, physical costs of adding links and limited capacity of a vertex [28,29] will limit the number of possible links attaching to a given vertex.

To test numerically the effect of aging and cost constraints on the local structure of networks with preferential attachment, we simulate the scale-free model of Ref. [5] but introduce aging and cost constraints of varying strength. Figure 4 shows that both types of constraints lead to cut-offs on the power-law decay of the tail of connectivity distribution and that for strong enough constraints no power-law region is visible.

We note that the possible distributions of connectivity of the small-world networks have an analogy in the theory of critical phenomena [30]. At the gas-liquid critical point, the distribution of sizes of the droplets of the gas (or of the liquid) is *scale-free*, as there is no free-energy cost in their formation [30]. As for the case of a scale-free network, the size $s$ of a droplet is power-law distributed: $p(s) \sim s^{-\alpha}$. As we move away from the critical point, the appearance of a non-negligible surface tension introduces a free-energy cost for droplets which limits their sizes, so that their distribution becomes *broad-scale*: $p(s) \sim s^{-\alpha} f(s/\xi)$, where $\xi$ is the typical size for which surface tension starts to be significant and the function $f(s/\xi)$ introduces a sharp cut-off for droplet sizes $s > \xi$. Far from the critical point, the scale $\xi$ becomes so small that no power-law regime is observed and the droplets become *single-scale* distributed: $p(s) \sim f(s/\xi)$. Often, the distribution of sizes in this regime is exponential or Gaussian.

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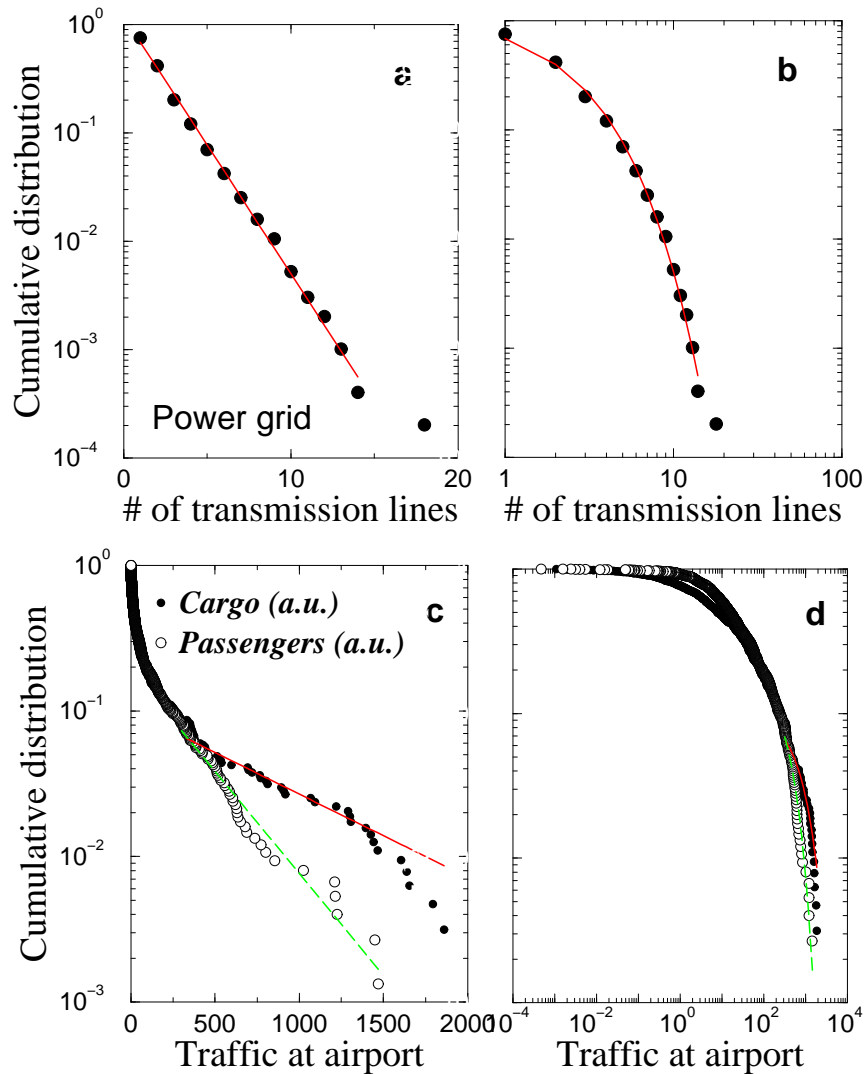
FIGURE 1



FIG. 1.  Technologic and economic networks. **a** Linear-log plot of the cumulative distribution of connectivities for the electric-power grid of Southern California [1].  For this type of plot, the distribution falls on a straight line indicating an exponential decay of the distribution of connectivities.  The full line, which is an exponential fit to the data, displays good agreement with the data. **b** Log-log plot of the cumulative distribution of connectivities for the electric-power grid of Southern California.  If the distribution would have a power law tail then it would fall on a straight line in a log-log plot.  Clearly, the data reject the hypothesis of power-law distribution for the connectivity. **c** Linear-log plot of the cumulative distribution of traffic at the world's largest airports for two measures of traffic, cargo and number of passengers.  The network of world airports is a small-world network: one can connect any two airports in the network by only 1-5 links.  To study the distribution of connectivities of this network, we assume that, for a given airport, cargo and number of passengers are proportional to the number of connections of that airport with other airports.  The data are consistent with a decay of the distribution of connectivities for the network of world airports that decays exponentially or faster.  The full line is an exponential fit to the cargo data for values of traffic between 500 and 1500.  For values of traffic larger than 1500, the distribution appears to decay even faster than an exponential.  The long-dashed line is an exponential to the passenger data for values of traffic between 500 and 1500. **d** Log-log plot of the cumulative distribution of traffic at the world's largest airports.  This plot confirms that the tails of the distributions decay faster than a power law would.
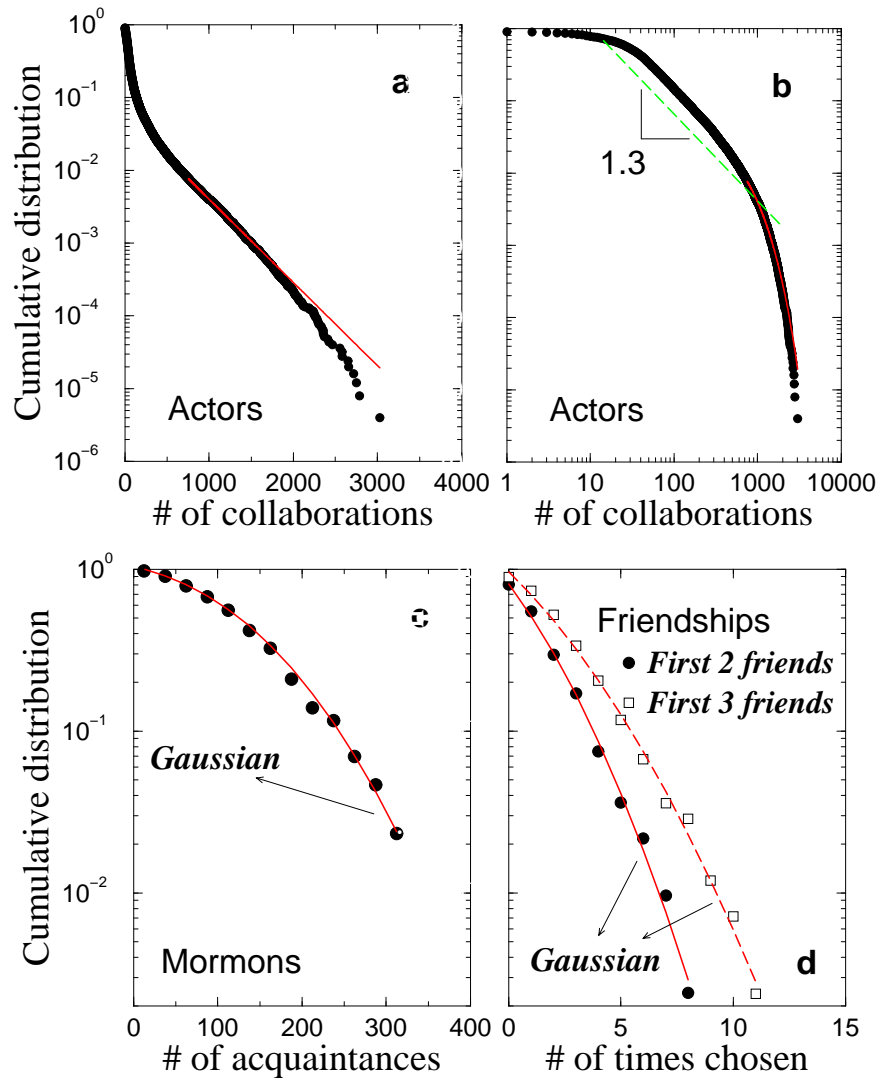
FIGURE 2



FIG. 2. Social networks. **a** Linear-log plot of the cumulative distribution of connectivities for the network of movie actors [1]. The full line is a guide for the eye of what an exponential decay would be. The data appear to fall faster in the tail than for an exponential decay, suggesting a Gaussian decay. Both exponential and Gaussian decays indicate that the connectivity distribution is not scale free. **b** Log-log plot of the cumulative distribution of connectivities for the network of movie actors. This plot suggests that for values of number of collaborations between 30 and 300 the data are consistent with a power-law decay. The apparent exponent of this cumulative distribution, $\alpha - 1 \approx 1.3$, is consistent with the value $\alpha = 2.3 \pm 0.1$ reported for the probability density function [5]. For larger numbers of collaborations the power-law decays is truncated. **c** Linear-log plot of the cumulative distribution of connectivities for the network of acquaintances of 43 Utah Mormons [23]. The full line is the fit to the cumulative distribution of a Gaussian. The tail of the distribution appears to fall off as a Gaussian, suggesting that there is a single scale for the number of acquaintances in social networks. **d** Linear-log plot of the cumulative distribution of connectivities for the friendship network of 417 high-school students [24]. The number of links is the number of times a student is chosen by another student as one of his/hers two (three) best friends. The lines are Gaussian fits to the empirical distributions.
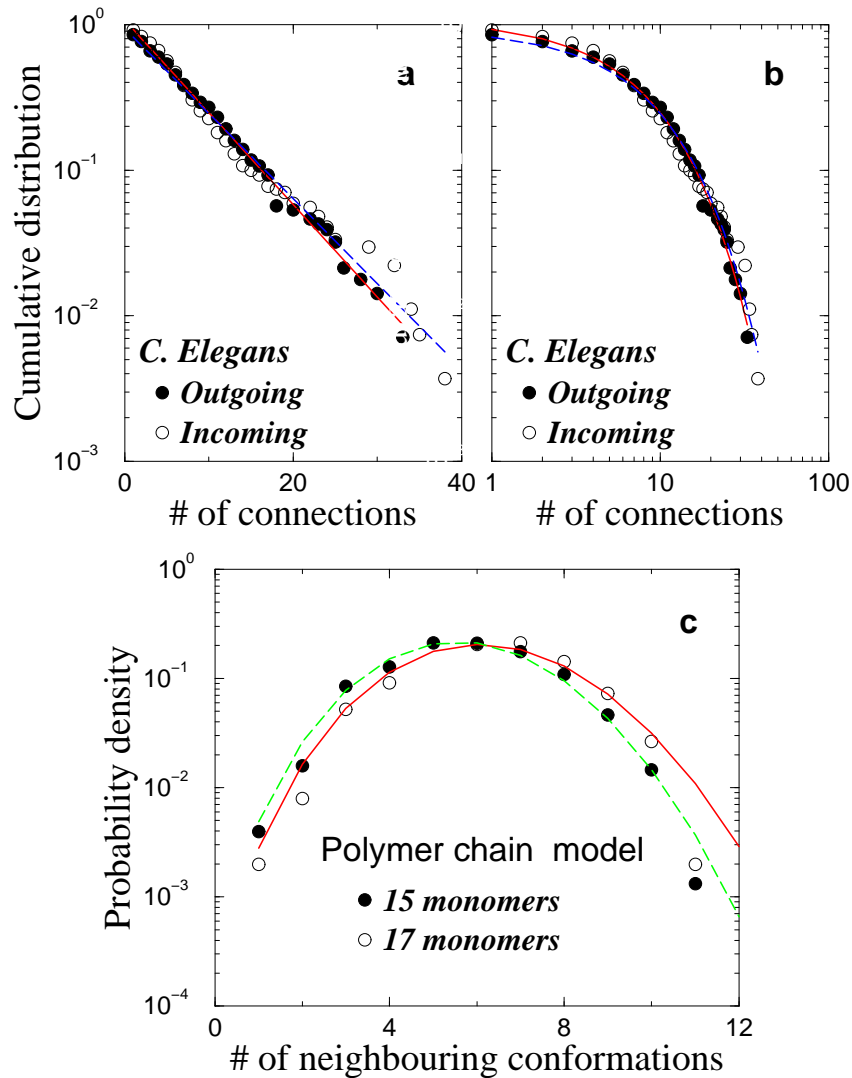
**FIGURE 3**



FIG. 3. Biologic and physical networks. **a** Linear-log plot of the cumulative distribution of outgoing and incoming connections for the neuronal network of the worm *C. Elegans* [25,26]. The full and long-dashed lines are exponential fits to the distributions of outgoing and incoming connections, respectively. The tails of the distributions appear to be consistent with an exponential decay. **b** Log-log plot of the cumulative distribution of outgoing and incoming connections for the neuronal network of the worm *C. Elegans*. If the distribution would have a power law tail then it would fall on a straight line in a log-log plot. The data appear to reject the hypothesis of a power-law distribution for the connectivity. **c** Linear-log plot of the probability density function of connectivities for the network of conformations of a lattice polymer chain [27]. A simple argument suggests that the connectivities follow a binomial distribution. The full and dashed lines are fits of a binomial probability density function to the data for polymer chains of different lengths. For the values of the parameters obtained in the fit, the binomial closely resembles the Gaussian indicating that there is a single scale for the connectivities of the conformation space of polymers.
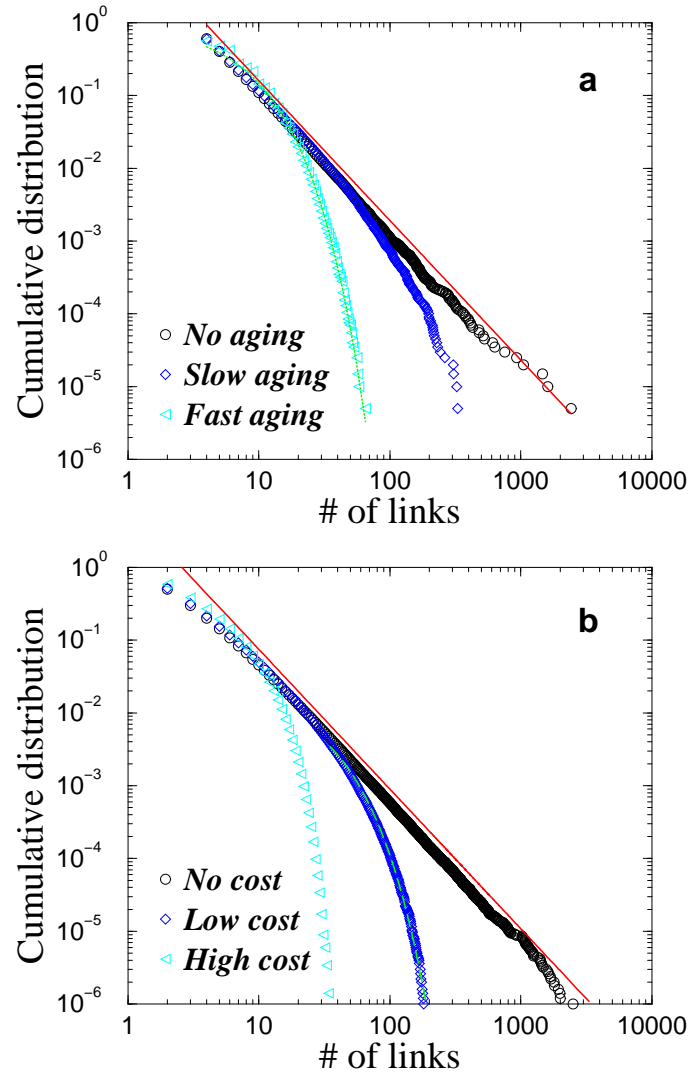
FIGURE 4



FIG. 4.  Truncation of scale-free connectivity by adding constraints to the model of Ref. [5]. **a** Effect of aging of vertices on the connectivity distribution; we see that aging leads to a cut-off of the power-law regime in the connectivity distribution. For sufficient aging of the vertices, the power-law regime disappears altogether. **b** Effect of cost of adding links on the connectivity distribution. Our results indicate that costs for adding links also leads to a cut-off of the power-law regime in the connectivity distribution and that for sufficient costs the power-law regime disappears altogether.